Section 17.1

σ-algebra: a collection of subsets of X, $\mathcal{F}$

1) $\phi \in \mathcal{F}$
2) closed under complements
3) " " countable unions.

Measurable space: $(X, \mathcal{F})$

(set function)

Measure: $\mu : \mathcal{F} \longrightarrow [0, \infty]$ st $\mu(\phi) = 0$

and $\mu\left(\bigsqcup_{k=1}^{\infty} E_k\right) = \sum \mu(E_k)$ (countable additivity)

Example: 1) $(\mathbb{R}, \text{Leb}, m)$ where Leb is the collection of Lebesgue meas sets.

2) $(\mathbb{R}, \mathcal{B}, m)$ ($\mathcal{B}$ = Borel meas. sets) smallest σ-algebra containing all open intervals.

3) $(X, 2^X, \eta)$ where 2^X = Power set,

η = counting measure $\eta(A) = |A|$

4) $(X, \mathcal{F}, \delta_{x_0})$ where $\delta_{x_0}(A) = 1$ if $x_0 \in A \in \mathcal{F}$

Called Dirac measure.

Restrictions of meas spaces. $(X, \mathcal{F}, \mu)$

and $X_0 \in \mathcal{F}$ Then

$(X_0, \mathcal{F}_0, \mu_0)$ is a measurable space.

where $\mathcal{F}_0 = \{A \cap X_0 \mid A \in \mathcal{F}\}$ $\mu_0 = \mu|_{\mathcal{F}_0}$

Prop 1. If $(X, \mathcal{F}, \mu)$ meas. space

1) Finite additivity
2) Monotonicity $\quad A \subset B$ then $\mu(A) \le \mu(B)$
3) Excision $A \subset B$ $\quad \mu(A) < \infty \Rightarrow \mu(B - A) = \mu(B) - \mu(A)$
4) for any $E \subset \bigcup_{k=1}^{\infty} E_k \qquad \mu(E) \le \sum_{k=1}^{\infty} \mu(E_k)$

Called countable monotonicity.

Pf: Same as Lebesgue meas.

Prop 2: (Continuity of meas.) $(X, \mathcal{F}, \mu)$. If $A_k \uparrow$, $B_k \downarrow$

1) $\{A_k\}_{k=1}^{\infty} \qquad \mu\left(\bigcup_k A_k\right) = \lim_{k\to\infty} \mu(A_k)$

don't state it at first

2) $\{B_k\}_{k=1}^{\infty} \qquad \mu\left(\bigcap_k B_k\right) = \lim_{k\to\infty} \mu(B_k)$ $\quad$ if $\mu(B_1) < \infty$

Pf:
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(A_1 \cup \bigsqcup_{k=1}^{\infty} A_{k+1} \setminus A_k\right)$$
$$= \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1}) - \mu(A_k) \quad \text{by definition.}$$

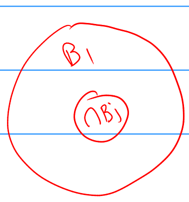
limit of partial sums
$$= \lim_{n\to\infty} \left(\mu(A_1) + \sum_{k=1}^{n} \mu(A_{k+1}) - \mu(A_k)\right)$$

By definition of the ∞ sum. This is a bit subtle
$$= \lim_{n\to\infty} \mu(A_n)$$

One should think of this as follows:
$$\lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} A_k\right) = \mu\left(\lim_{n\to\infty} \bigcup_{k=1}^{n} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

b) For the 2nd part, write

countable additivity
$$\mu\left(B_1 \setminus \bigcap_{j=1}^{\infty} B_j\right) = \mu\left(\bigsqcup_{k=1}^{\infty} B_k \setminus B_{k+1}\right) = \lim_{n\to\infty} \mu(B_1) - \mu(B_{n+1})$$
$$\Vert$$
$$\mu(B_1) - \mu\left(\bigcap_{j=1}^{\infty} B_j\right)$$



In class break

Ex: $B_j = [j, \infty)$ and

Borel Cantelli lemma: Let $(X, \mathcal{F}, \mu)$ meas space.

$\{E_n\}$ st $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then $\mu(E_n \text{ i.o.}) = 0$

$$\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_k\right) = 0$$

The set of all points in X that belong to infinitely many E_k has measure 0.

Innocuous looking, but very valuable!

Good place to recall the Indicator function of a set.

$$1_E(x) \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

$$\overline{\lim}\, 1_{E_k}(x) = 1 \iff \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_m \ni x$$

So we write $\overline{\lim}\, E_k := \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_m$

Pf: $\mu\left(\bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} E_m\right) \leq \mu\left(\bigcup_{m=k}^{\infty} E_m\right)$ for any k

$\leq \sum_{m=k}^{\infty} \mu(E_m)$

σ-finite and finite spaces: μ is finite if $\mu(X) < \infty$

μ is σ-finite if $\exists \{E_k\}_{k=1}^{\infty}$ $\bigcup_k E_k = X$ and $\mu(E_k) < \infty$

We can use the same terminology (finite and σ-finite) for SUBSETS of X as well.

Remark: Cover can be taken to be disjoint.

Prop: (Complete spaces) $(X, \mathcal{F}, \mu)$ is complete if it contains all subsets of measure 0 sets. That is if $E \subset A \in \mathcal{F}$ st $\mu(A) = 0$, then $E \in \mathcal{F}$.

(All measure spaces can be completed)

Prop Let $(X, \mathcal{F}, \mu)$ be a meas space. Let $\mathcal{F}_0$ be $E \subset X$ st $E = A \cup B$, where A is a subset of a meas 0 set and $B \in \mathcal{F}$.

Define $\mu_0(E) = \mu(B)$

Then $\mathcal{F}_0$ contains $\mathcal{F}$

$\mu_0|_{\mathcal{F}} = \mu$ and

$(X, \mathcal{F}_0, \mu_0)$ is a complete meas. space.

Ex: $(\mathbb{R}, \mathcal{B}, leb)$ $\hookrightarrow$ Borel sets $(\mathbb{R}, \tilde{\mathcal{F}}, leb)$ $\uparrow$ Lebesgue sets.

☆ good exercise

Pf: $\mathcal{F}_0 \supset \mathcal{F}$ obvious μ_0 is a measure: $\mu_0(\phi \cup \phi) = \mu(\phi) = 0$

$\mu_0\left(\bigsqcup_{i=1}^{\infty} A_i \cup B_i\right)$ $A_i \cup B_i \cap A_j \cup B_j = \phi \Rightarrow A_i \cap A_j = \phi$

$= \mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

Take $E = A \cup B$ st $\mu_0(E) = \mu(A) = 0$

So $E \subset A \cup B \subset D$ where $\mu(D) = 0$

$\uparrow$ this is contained in a measure 0 set!

So for any $F \subset E$ is also contained in a meas. 0 set.